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A TAUBER TYPE THEOREM FOR SERIES IN BESSEL FUNCTIONS

Jordanka Paneva-Konovska

Abstract

Some important properties of the power series are given by the classical Cauchy-Hadamard, Abel and Tauber theorems. For series in Bessel functions of first kind $\sum\limits_{n=0}^{\infty} a_n J_n(z)$, the corresponding Cauchy-Hadamard and Abel theorems have been proved by the author in a previous paper [2]. In this paper we prove a Tauber type theorem for series in Bessel functions of first kind.

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Key Words and Phrases: Bessel functions, Abelian and Tauberian theorems for power series

1. Introduction

Some important properties of the power series $\sum\limits_{n=0}^{\infty}$ are given by the classical Cauchy-Hadamard, Abel and Tauber theorems.

In general, by the classical Abel theorem, from the convergence of the power series $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$ at a point z_0 it follows the existence of the limit $\lim\limits_{z \to z_0} f(z) = f(z_0)$ when z belongs to a suitable angle domain with a vertex at a point z_0. The geometrical series [4, p.92]: $\dfrac{1}{1+z} = 1 - z + z^2 - z^3 + \dots \; (z_0 = 1)$ shows that the inverse proposition in general is not true, i.e. the existence of this limit does not imply the convergence of the series $\sum\limits_{n=0}^{\infty} a_n z^n$ without additional conditions on the growth of the coefficients.

The corresponding result in this direction is given by the following theorem.

Theorem (Tauber). *If the coefficients of the power series satisfy the condition* $\lim\limits_{n\to\infty} na_n = 0$ *and if* $\lim\limits_{z\to 1} f(z) = S$ *($z \to 1$ radially), then the series* $\sum a_n$ *is convergent and* $\sum\limits_{n=0}^{\infty} a_n = S$.

It turns out that the Abel theorem fails even for series of the kind $\sum\limits_{k=1}^{\infty} a_{n_k} z^{n_k}$, where $(n_1, n_2, \ldots, n_k, \ldots)$ is a suitable permutation of the nonnegative integers [4, p.92]. Therefore, it is interesting to know if for series in a given sequence of holomorphic functions a statement like the Abel theorem is available. A positive answer to this question for series in Laguerre and Hermite polynomials is given by P. Rusev [3, §11.3] and L. Boyadjiev [1].

Let $J_n(z)$ $(n = 0, 1, 2, \ldots)$ be the Bessel functions of first kind. Let us consider the series of the form $\sum\limits_{n=0}^{\infty} a_n J_n(z)$. In our previous paper [2] corresponding Cauchy-Hadamard and Abel theorems have been proved.

In this paper we prove the corresponding Tauber type theorem for series in Bessel functions of first kind.

2. A Tauber type theorem

Let us consider the series $\sum\limits_{n=0}^{\infty} a_n$, $a_n \in \mathbb{C}$. Let $z_0 \in \mathbb{C} \setminus \mathbb{R}, |z_0| = R$. For shortness, denote

$$J_n^*(z; z_0) = \frac{J_n(z)}{J_n(z_0)}.$$

Let the series

$$F(z) = \sum_{n=0}^{\infty} a_n J_n^*(z; z_0)$$

be convergent for $|z| < R$. Then next theorem is valid.

Theorem (Tauber type). *If there exists*

$$\lim_{z\to z_0} F(z) = S \quad (|z| < R, z \to z_0 \text{ radially})$$

and

$$\lim_{n\to\infty} na_n = 0, \tag{1}$$

then the series $\sum\limits_{n=0}^{\infty} a_n$ *is convergent and*

$$\sum_{n=0}^{\infty} a_n = S.$$

P r o o f. For a point z of the segment $[o, z_0]$ we have

$$\sum_{n=0}^{k} a_n - F(z) = \sum_{n=0}^{k} a_n - \sum_{n=0}^{\infty} a_n J_n^*(z; z_0)$$

$$= \sum_{n=0}^{k} a_n \frac{J_n(z_0)}{J_n(z_0)} - \sum_{n=0}^{\infty} a_n \frac{J_n(z)}{J_n(z_0)} = \sum_{n=0}^{k} a_n \frac{J_n(z_0) - J_n(z)}{J_n(z_0)} - \sum_{n=k+1}^{\infty} a_n \frac{J_n(z)}{J_n(z_0)}$$

and therefore,

$$|\sum_{n=0}^{k} a_n - F(z)| \leq \sum_{n=0}^{k} |a_n| \left| \frac{J_n(z_0) - J_n(z)}{J_n(z_0)} \right| + \sum n = k + 1^{\infty} |a_n| \left| \frac{J_n(z)}{J_n(z_0)} \right|. \quad (2)$$

By using the asymptotic formula [5, §17.81]

$$J_n(z) = \left(\frac{z}{2}\right)^n (1 + \theta_n(z)) \frac{1}{n!}, \theta_n(z) \to 0 \text{ as } n \to \infty \quad (3)$$

for the Bessel functions of first kind, we obtain:

$$a_n \frac{J_n(z)}{J_n(z_0)} = a_n \left(\frac{z}{z_0}\right)^n \frac{1 + \theta + n(z)}{1 + \theta_n(z_0)} = a_n \left(\frac{z}{z_0}\right)^n \left(1 + \widetilde{\theta}_n(z; z_0)\right).$$

Let ε be an arbitrary positive number. We choose a number N_1 so large that the inequalities $|1 + \widetilde{\theta}_k(z; z_0)| < 2, |k a_k| < \frac{\varepsilon}{6}$ hold as $k \geq N_1$. If $k > N_1$ and z is on the segment $[0, z_0]$, then for the second summand in (2) the following estimate is valid:

$$\begin{aligned}
\sum_{n=k+1}^{\infty} |a_n| \left| \frac{J_n(z)}{J_n(z_0)} \right| &= \sum_{n=k+1}^{\infty} |a_n| \left| \frac{z}{z_0} \right|^n |1 + \widetilde{\theta}_n(z; z_0)| \\
\leq 2 \left| \frac{z}{z_0} \right|^{k+1} \sum_{n=k+1}^{\infty} |a_n| \left| \frac{z}{z_0} \right|^{n-k-1} &\leq \sum_{n=0}^{\infty} |a_{n+k+1}| \left| \frac{z}{z_0} \right|^n \qquad (4) \\
= 2 \sum_{n=0}^{\infty} \frac{|(n+k+1) a_{n+k+1}|}{n+k+1} \left| \frac{z}{z_0} \right|^n &< 2 \sum_{n=0}^{\infty} \frac{\varepsilon/6}{n+k+1} \left| \frac{z}{z_0} \right|^n \\
< \frac{2}{k} \frac{\varepsilon}{6} \frac{1}{1 - |z/z_0|} &= \frac{\varepsilon}{3} \frac{1}{k} \frac{|z_0|}{|z_0| - |z|}.
\end{aligned}$$

Now let us consider the first summand in (2). We have:

$$\sum_{n=0}^{k} |a_n| \left| \frac{J_n(z_0) - J_n(z)}{J_n(z_0)} \right| = \sum_{n=0}^{m} |a_n| \left| \frac{J_n(z_0) - J_n(z)}{J_n(z_0)} \right| + \sum_{n=m+1}^{k} |a_n|$$

According to Schwarz's lemma, there exists a constant C such that

$$\left|\frac{J_n(z_0)-J_n(z)}{J_n(z_0)}\right| < C|z-z_0|.$$

Moreover, there exists a number N_2 such that the following inequality

$$\sum_{n=0}^{m}|a_n|\left|\frac{J_n(z_0)-J_n(z)}{J_n(z_0)}\right| \le C\,|z-z_0|\,k\,\frac{\sum\limits_{n=0}^{m}|a_n|}{k} < C\,|z-z_0|\,k\,\frac{\varepsilon}{3RC} = |z-z_0|\,k\,\frac{\varepsilon}{3R}. \tag{5}$$

holds as $k > N_2$. It remains to estimate the sum

$$\sum_{n=m+1}^{k}|a_n|\left|\frac{J_n(z_0)-J_n(z)}{J_n(z_0)}\right|.$$

To this end, using asymptotic formula (3) for the Bessel functions of first kind, we find consequently:

$$\begin{aligned}\frac{J_n(z_0)-J_n(z)}{J_n(z_0)} &= \frac{(z_0)^n(1+\theta_n(z_0))-z^n(1+\theta_n(z))}{z_0^n(1+\theta_n(z_0))}\\ = 1-\left(\frac{z}{z_0}\right)^n\frac{1+\theta_n(z)}{1+\theta_n(z_0)} &= 1-\left(\frac{z}{z_0}\right)^n\left[1+\frac{\theta_n(z)-\theta_n(z_0)}{1+\theta_n(z_0)}\right]\\ &= 1-\left(\frac{z}{z_0}\right)^n-\left(\frac{z}{z_0}\right)^n\frac{\theta_n(z)-\theta_n(z_0)}{1+\theta_n(z_0)}\,.\end{aligned}$$

Therefore,

$$\left|\frac{J_n(z_0)-J_n(z)}{J_n(z_0)}\right| \le \left|1-\left(\frac{z}{z_0}\right)^n\right| + \left|\frac{z}{z_0}\right|^n\left|\frac{\theta_n(z)-\theta_n(z_0)}{1+\theta_n(z_0)}\right|. \tag{6}$$

We obtain the following inequalities

$$\left|1-\left(\frac{z}{z_0}\right)^n\right| = \left|1-\frac{z}{z_0}\right|\left|1+\frac{z}{z_0}+\left(\frac{z}{z_0}\right)^2+\ldots+\left(\frac{z}{z_0}\right)^{n-1}\right| \le n\left|1-\frac{z}{z_0}\right|$$

for the first summand of (6). According to Schwarz's lemma, there exists a constant ρ such that

$$\left|\frac{\theta_n(z)-\theta_n(z_0)}{1+\theta_n(z_0)}\right| \le 1 \qquad \text{as} \qquad |z-z_0|<\rho.$$

Then, for such $|z|$, we obtain for the second summand of (6):

$$\left|\frac{z}{z_0}\right|^n\left|\frac{\theta_n(z)-\theta_n(z_0)}{1+\theta_n(z_0)}\right| \le \left|\frac{z}{z_0}\right|^n |z-z_0|.$$

From (1) it follows that

$$\lim_{n\to\infty} a_n = 0, \quad \lim_{k\to\infty} \frac{\sum\limits_{n=1}^{k} n|a_n|}{k} = 0, \quad \lim_{k\to\infty} \frac{\sum\limits_{n=1}^{k} |a_n|}{k} = 0.$$

Then a number N_3 exists such that

$$\frac{\sum\limits_{n=m+1}^{k} n|a_n|}{k} < \frac{\varepsilon}{3(1+R)} \qquad \text{and} \qquad \frac{\sum\limits_{n=m+1}^{k} |a_n|}{k} < \frac{\varepsilon}{3(1+R)} \qquad \text{as} \qquad k > N_3.$$

Therefore,

$$\begin{aligned} \sum_{n=m+1}^{k} |a_n| \left|\frac{J_n(z_0)-J_n(z)}{J_n(z_0)}\right| &\le \sum_{n=m+1}^{k} n|a_n| \left|1-\frac{z}{z_0}\right| + \sum_{n=m+1}^{k} |a_n| \left|\frac{z}{z_0}\right|^n |z-z_0| \\ &\le k\,\frac{|z-z_0|}{R}\,\frac{\sum\limits_{n=m+1}^{k} n|a_n|}{k} + k\,|z-z_0|\,\frac{\sum\limits_{n=m+1}^{k} |a_n|}{k} \qquad (7) \\ &< k\,|z-z_0|\,\frac{1+R}{R}\,\frac{\varepsilon}{3}(1+R) = k\,|z-z_0|\,\frac{\varepsilon}{3R}. \end{aligned}$$

Finally, let us note that

$$\begin{aligned} \left|\sum_{n=0}^{k} a_n - F(z)\right| &\le \sum_{n=0}^{m} |a_n| \left|\frac{J_n(z_0)-J_n(z)}{J_n(z_0)}\right| \\ &+ \sum_{n=m+1}^{k} |a_n| \left|\frac{J_n(z_0)-J_n(z)}{J_n(z_0)}\right| + \sum_{n=k+1}^{\infty} |a_n| \left|\frac{J_n(z)}{J_n(z_0)}\right|. \end{aligned}$$

Let $N = \max(N_1, N_2, N_3), k > N$ and $|z-z_0| < \rho$. Then by using (4),(5),(7), we can conclude that

$$\begin{aligned} \left|\sum_{n=0}^{k} a_n - F(z)\right| &< |z-z_0|\,k\,\frac{\varepsilon}{3R} + k\,|z-z_0|\,\frac{\varepsilon}{3R} + \frac{\varepsilon}{3}\,\frac{1}{k}\,\frac{|z_0|}{|z_0|-|z|} \\ &= \frac{\varepsilon}{3}\left[\frac{2k}{R}\,|z-z_0| + \frac{1}{k}\,\frac{|z_0|}{|z_0|-|z|}\right]. \end{aligned}$$

If we substitute z by $z_0(1-\frac{1}{k})$, then

$$\left|\sum_{n=0}^{k} a_n - F\left(z_0(1-\frac{1}{k})\right)\right| < \frac{\varepsilon}{3}\,3 = \varepsilon.$$

This proves that $\lim\limits_{k\to\infty} \sum\limits_{n=0}^{k} a_n$ exists and equals $\lim\limits_{k\to\infty} F\left(z_0(1-\frac{1}{k})\right)$, i.e.

$$\sum_{n=0}^{\infty} a_n = \lim_{k\to\infty} F\left(z_0(1-\frac{1}{k})\right) = S.$$

Thus the theorem is proved. ■

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Institute of Applied Mathematics and Informatics
Technical University of Sofia
Sofia 1156, BULGARIA

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